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Transient Chaos and Critical States in Generalized Baker Maps

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Generalized multibaker maps are introduced to model dissipative systems which are spatially extended only in certain directions and escape of particles is allowed in other ones. Effects of nonlinearity are investigated by varying a control parameter. Emphasis is put on the appearance of the critical state representing the borderline of transient chaos, where anomalous behavior sets in. The investigations extend to the conditionally invariant and the related natural measures and to transient diffusion in normal and critical states as well. Permanent chaos is also considered as a special case.

KEY WORDS: Transient chaos; conditionally invariant measures; natural measures; critical state; diffusion.

1. INTRODUCTION

Transient chaos has become of equal importance as the permanent one in the last decade. Its field of applications has included chaotic scattering, chaotic advection and transport phenomena.[1–9] The conditionally invariant measure[10,6] and the related natural one[11,6] are the counterparts of the invariant measure existing in systems exhibiting permanent chaos. Statistical properties of the trajectories during their chaotic development can be expressed with the help of these measures. In particular the properties of diffusion in systems whose trajectories can escape from the phase space considered is governed by the natural measure. The main purpose of the present paper is to further explore the features of measures attached to transient chaos in dissipative systems in particular to follow their change

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when a control parameter is varied up to the borderline where the transient chaos shows critical behavior. The motivation of the present paper stems from the features of deterministic diffusion in transiently chaotic systems.[7] Its peculiar properties can be traced back to those of the conditionally invariant measures and the related natural ones. In this problem two main areas of chaotic phenomena are present simultaneously, namely transient chaos and deterministic diffusion, which has played a central role in understanding macroscopic transport properties from the point of view of deterministic microscopic motion since the pioneering work by Gaspard and Nicolis.[1]

Deterministic diffusion has mainly been studied within the framework of two models, namely considering an infinite chain of one-dimensional maps[12–16, 1, 4, 7] or with the help of multibaker maps.[2, 3, 5] In such systems due to translational invariance the statistical properties of chaos can be investigated within the unit cell by introducing the reduced map. In case of the dyadic multibaker map[3] the reduced map agrees with the original Baker map. Needless to say that the reduced map can be interesting in itself as a two-dimensional map exhibiting various chaotic behavior independently from its relationship to an infinite chain of similar maps.

In our previous paper we extended models of the first type to allow escape at each step, which results in a reduced map with a window, where the trajectory can leave the system.[7] To study transient diffusion further the present paper generalizes the dyadic multibaker map in two respects. Namely, we introduce again a window in the reduced map and furthermore treat nonlinear two-dimensional maps instead of the piecewise linear baker map. The latter extension makes it possible to follow the parameter dependence of characteristics from the baker map up to the borderline situation, where hyperbolicity is violated and the limiting value of the escape rate agrees with the positive Lyapunov exponent of the saddle point at the origin. Since we want to study dissipative systems, the maps treated are invertible but not invariant under time-reversal; i.e., taking the opposite time direction we do not get the same map or its conjugate. (See ref. 3 for general definition of time-reversal invariance and for the proof that the original multibaker map fulfills the requirements. A time-reversible generalization of the Baker map is given in ref. 8.)

The reduced maps considered have conditionally invariant measures which are smooth along the unstable manifold. It is pointed out that while the conditionally invariant measures[10, 17, 18] are different for forward and backwards iterations, the related natural measures are the same, which ensures that the diffusion coefficient has the same value in both directions. Approaching the borderline situation[19, 20] the natural measure (but not the conditionally invariant one) degenerates to a Dirac delta function located at the origin, which has the consequence that the diffusion coefficient tends to zero in this limit. At the borderline situation, however, further conditionally invariant measures smooth in the unstable direction appear, which have, of coarse, different basins of attraction. The type of the latter ones may generate a non-degenerate natural measure and a finite diffusion coefficient in one of the directions of iterations.

In the permanently chaotic case anomalous diffusion[15, 16, 21, 22] can occur in the strongly intermittent state (when the reduced map with the considered measure has Kolmogorov–Sinai entropy $K=0$). Following the borderline situations, where the transient chaos is critical the limit when the escape goes to zero can lead to weakly ($K>0$) or strongly intermittent dynamics. In the former case there remains no anomalous behavior in the diffusion.

The paper is organized as follows. In Section 2 the model to be investigated is introduced and its properties are discussed. The invariant set (chaotic saddle) of the reduced maps is specified, and the Frobenius–Perron operator, the conditionally invariant measure, and the natural measure are defined and studied in Section 3. The special case of the critical (borderline) situation is treated in Section 4. Properties of transient diffusion are considered in Section 5. Section 6 is devoted to a discussion.

2. GENERALIZED MULTIBAKER MAPS

We introduce the two-dimensional maps $(x', y', S')=F(x, y, S)$ with the specification

$$F(x, y, S)=\begin{cases}(f(x), J^F y/|f'(x)|, S-1) & \text{if} \quad x\in \mathrm{I}_0\\ (f(x), 1-J^F y/|f'(x)|, S+1) & \text{if} \quad x\in \mathrm{I}_1\end{cases} \tag{1}$$

where S is an integer variable, $f(x)$ is a smooth function mapping $[0,1]$ twice onto itself, $\mathrm{I}_0=f_l^{-1}[0, 1]$, $\mathrm{I}_1=f_u^{-1}[0, 1]$ and f_l^{-1}, f_u^{-1} denote the lower and upper branches of the inverse of $f(x)$, respectively. We shall assume that $f(x)$ is increasing in I_0 and decreasing in I_1. It may have an escape window between I_0 and I_1, for which it is not defined.

The map (1) has a constant Jacobian $J^F>0$. It acts on an infinite chain of unit squares. S labels the square which is visited by the particle and (x, y) determines the point inside the square. During the mapping the particle jumps to the square to the right if $x\in \mathrm{I}_1$ or to the left if $x\in \mathrm{I}_0$. If there is an escape window in f then there is a third possibility, the particle leaves the system if $x\in[0, 1]\backslash \mathrm{I}_0\backslash \mathrm{I}_1$.

Translational invariance of the system makes it possible to use the so called reduced map for a partial description of the system.[12–14] It can be obtained by considering the cells to be identical, and thereby, following only the motion inside one cell. The reduced map $F(x, y)$ is the same as $F(x, y, S)$ disregarding the variable S.

The reduced map of (1) maps two rectangles of the unit square $U = [0, 1] \times [0, 1]$ in the (x, y) plane to two regions (see Fig. 1), which can in general overlap. To exclude this overlap, which causes problem concerning invertibility, we choose the 1D map $f(x)$ such, that the constant density $P(x) \equiv 1$ be conditionally invariant under its action. $P(x)$ satisfies the one-dimensional Frobenius–Perron equation

$$e^{-\kappa} P(x) = \frac{P(f_l^{-1}(x))}{|f'(f_l^{-1}(x))|} + \frac{P(f_u^{-1}(x))}{|f'(f_u^{-1}(x))|} \tag{2}$$

where κ is the escape rate. Requiring that $P(x) \equiv 1$ be a solution of Eq. (2) by integration and taking into account the properties of $f(x)$ listed below Eq. (1) one obtains

$$f_u^{-1}(x) - f_l^{-1}(x) = 1 - e^{-\kappa} x \tag{3}$$

In the following it will be understood that $f(x)$ satisfies this requirement. Note that such a condition does not restrict generality in the choice of the 1D map, since any map $f(x)$, as specified below (1), with smooth conditionally invariant measure can be transformed to a map with constant conditionally invariant density.[23, 19] The necessary transformation is a conjugation $\mu(f(\mu^{-1}(x)))$, where $\mu(x)$ is the conditionally invariant measure of $[0, x]$ for the map $f(x)$.

The general form of such a map can be given with the formulas for its inverse[23, 19] as follows

$$f_l^{-1}(x) = \frac{x}{2R} + v\left(\frac{x}{2R}\right) \tag{4}$$

$$f_u^{-1}(x) = 1 - \frac{x}{2R} + v\left(\frac{x}{2R}\right) \tag{5}$$

Here $v(x)$ is a smooth function with properties $v(0) = 0$ and $|v'(x)| \leqslant 1$, where equality is allowed only in isolated points. Substituting (4), (5) into (2) one obtains $R = e^{\kappa}$. Note that $v = 0$ leads to the tent map. With the choice $v(x) = d \cdot x(1 - x)$, $d \in [-1, 1]$ Eqs. (4), (5) specify a one-dimensional map which is conjugated to the one used in ref. 7.

To obtain condition for invertibility of F let us determine the preimage of a vertical line segment $V_x = \{(x, y) \mid y \in [0, 1]\}$. It is reached from two vertical segments V_{x_l} and V_{x_u}. See Fig. 1, where the segment of V_x which does not have a preimage is shown by dotted line. The corresponding x values are $x_l = f_l^{-1}(x)$ and $x_u = f_u^{-1}(x)$, respectively. The first segment, V_{x_l} is mapped to the subinterval $y \in [0, J^F/f'(f_l^{-1}(x))]$, and V_{x_u} to $y \in [1 - J^F/|f'(f_u^{-1}(x))|, 1]$ on V_x. From (2) follows, that these subintervals do not overlap, if

$$J^F e^{-\kappa} \leqslant 1 \tag{6}$$

where κ is defined by (2) with $P(x) \equiv 1$. In the following this condition shall be assumed to be satisfied.

As x runs through the interval $[0, 1]$ the subintervals mentioned above fill in two separate regions, namely the images FA_0, FA_1 of $A_0 = I_0 \times [0, 1]$ and $A_1 = I_1 \times [0, 1]$ (see Fig. 1). A special feature of the map with $v(x) = dx(1-x)$ is, that the boundaries of FA_0 and FA_1 are straight lines. However, further iterates have curved boundaries for $d \neq 0$ even in this case. On the other hand the length of the gap between the two intervals $[0, J^F/f'(f_l^{-1}(x))]$ and $[1 - J^F/|f'(f_u^{-1}(x))|, 1]$ is $1 - J^F e^{-\kappa}$, which is independent of x, yielding the occurrence of an empty stripe in the unit square with constant vertical thickness if $J^F e^{-\kappa} < 1$. Even in the limiting case when the thickness becomes zero (i.e., equality holds in (6)) the common points of FA_0 and FA_1 form a set of measure zero, such as in the case of the original baker map, and therefore they can be disregarded. Thereby the reduced map, and also (1) are invertible for typical points. The inverse map reads

$$G(x, y, S) = \begin{cases} (f_l^{-1}(x), J^G f'(f_l^{-1}(x))\, y, S+1) \\ \qquad \text{if} \quad y \leqslant \dfrac{1}{J^G f'(f_l^{-1}(x))} \\ (f_u^{-1}(x), J^G\, |f'(f_u^{-1}(x))|\, (1-y), S-1) \\ \qquad \text{if} \quad y \geqslant 1 - \dfrac{1}{J^G\, |f'(f_u^{-1}(x))|} \end{cases} \tag{7}$$

where $J^G = (J^F)^{-1}$ is the Jacobian of the iteration G. Similarly to F, G maps points of a square to the neighboring cells. Namely, the point jumps to the right if $(x, y) \in FA_0$ and to the left if $(x, y) \in FA_1$.

For the iteration F the range $J^F \in (0, 1)$ is the physically interesting one. However, if we consider the iteration G to be the physical direction, then $J^G \in (0, 1)$ is the relevant one. While in case of iteration F one of the variables, namely x, transforms independently from the other one, it is no more true in case of the iteration G. This means that the two-dimensional

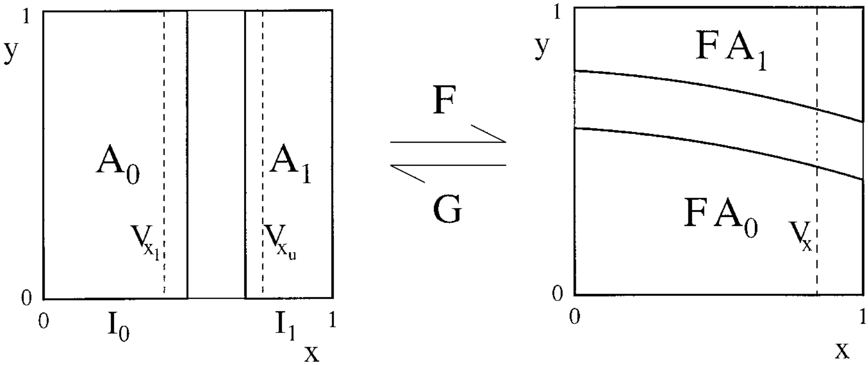


Fig. 1. Schematic drawing of the action of the reduced maps F and G.

features of the map are more pronounced in the latter case. We consider F and G on equal footing.

The maps (1) and (7) have the advantage that their reduced maps, which are generalized baker maps, have complete grammar if one chooses the symbolic dynamics generated by the partition (A_0, A_1). We note, that similar maps have been studied in ref. 24 in other context.

Equation (1) contains as special cases piecewise linear maps by choosing $f(x) = ax$ if $x < 1/a$, and $f(x) = a(1-x)$ if $x \geqslant 1 - 1/a$, or, alternatively, $f(x) = ax$ if $x < 1/a$ and $f(x) = 1 - a(1-x)$ if $x \geqslant 1 - 1/a$. The latter one with the choice $a = 2$ gives back the original dyadic multibaker map.[3]

3. THE INVARIANT SET AND INVARIANT MEASURES

To discuss invariant measures one has to study evolution equation of a probability distribution under the effect of the reduced map. The Frobenius–Perron operator transforming the density of a smooth measure in two dimensions can be written for the map $F(x, y)$ as

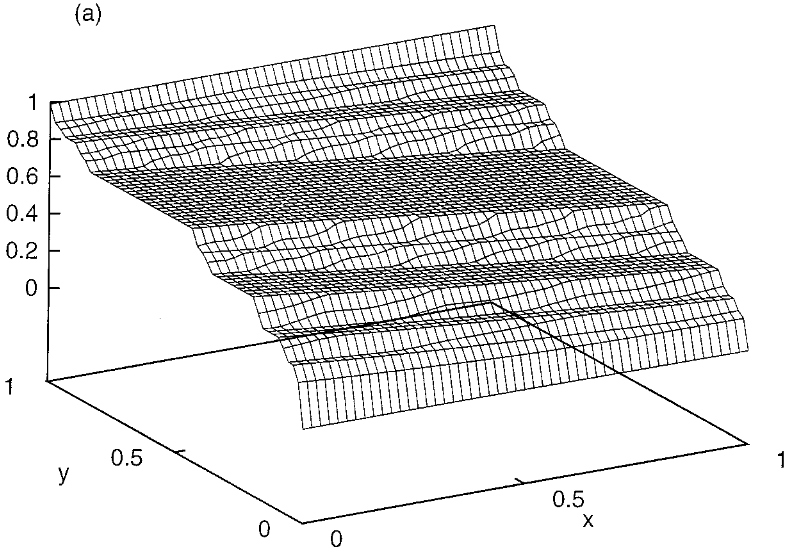
$$LP(x, y) \equiv \begin{cases} P(F^{-1}(x, y))/J^F & \text{if} \quad F^{-1}(x, y) \in \mathrm{U} \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

with $\mathrm{U} = [0, 1] \times [0, 1]$ and similarly for the map G.

Consider first the map F with escape. The conditionally invariant measure μ^F is defined as the limit of the measures with densities $\exp(\kappa^F n)\, L^n P_0(x, y)$ when $n \to \infty$, where κ^F is the escape rate related to F.

It is easy to see, that in the unstable direction the conditionally invariant measure is smooth. Since iteration of x by the reduced map is independent of y, the projection of the conditionally invariant measure to the x axis is equal to the conditionally invariant measure of the 1D map, which has constant density. It also follows that the corresponding escape rate κ^F equals to the escape rate κ of the 1D map f. On the other hand, from the appearance of the empty stripe it is obvious, that the conditionally invariant measure has a fractal basis in the y direction, if $J^F \exp(-\kappa^F) < 1$. Consequently it does not have a smooth density and can be best demonstrated by derivating it only in x direction, i.e., by displaying $\mu_x^F(x, y) = (\partial/\partial x)\, \mu^F([0, x] \times [0, y])$. When there is a density it is connected to μ_x^F as $P^F(x, y) = (\partial/\partial y)\, \mu_x^F(x, y)$. However, in the present case μ_x^F is a devil's staircase in three dimensions, as seen in Fig. 2a in a finite resolution, and P^F is a bounded function only in numerical approximation. In area preserving case there should be an escape to fulfill this inequality, while in the dissipative case ($J^F < 1$) escape is not necessary. In case of area expanding maps (when G is physically relevant) $\kappa^F > \log J^F$ should stand. In the case $J^F \exp(-\kappa^F) = 1$ starting from a uniform density $P_0(x, y) = 1$ the full square is filled uniformly after one mapping, because of the constant Jacobi determinant. Then there are still two possibilities. If $\kappa^F > 0$ the resulting distribution gives back P_0 after normalization, so the conditionally invariant measure is the Lebesgue measure. If $\kappa^F = 0$ then $J^F = 1$ and the situation is the same but normalization is not necessary.

The natural measure is defined by the distribution of points of trajectories that have been started n iterations before according to an initial distribution with density $P_0(x, y)$ and stay in the system for at least another n iterations, also taking the limit $n \to \infty$. The first n steps map $P_0(x, y)$ to a smooth distribution in $F^n \mathrm{U} \cap \mathrm{U}$, thereby approaching the conditionally invariant measure when normalized. The restriction for the further n steps selects the points in $F^{-n}\mathrm{U}$. Since points that have left U are assumed not to return to it, the common part $\mathrm{U}_n = F^n \mathrm{U} \cap F^{-n} \mathrm{U}$ lies inside U, and gives the invariant set in the limit. The above restrictions mean that the natural measure is obtained as the limit of the iterated and normalized distribution in $\mathrm{U}_n = F^n \mathrm{U} \cap F^{-n} \mathrm{U}$. In the rest of this section we assume that the initial distribution is according to the Lebesgue measure. Because the Jacobian is constant the result of n steps is the Lebesgue measure restricted to U_n (apart from normalization), and the natural measure is its limit for $n \to \infty$. The invariant set and the natural measure are fractal in both directions if $J^F \exp(-\kappa^F) < 1$ and $\kappa^F > 0$, since the first condition ensures presence of horizontal stripes in $F^n \mathrm{U}$ and the latter creates vertical strips in $F^{-n}\mathrm{U}$. An exception is the critical situation, which will be discussed in the next section.



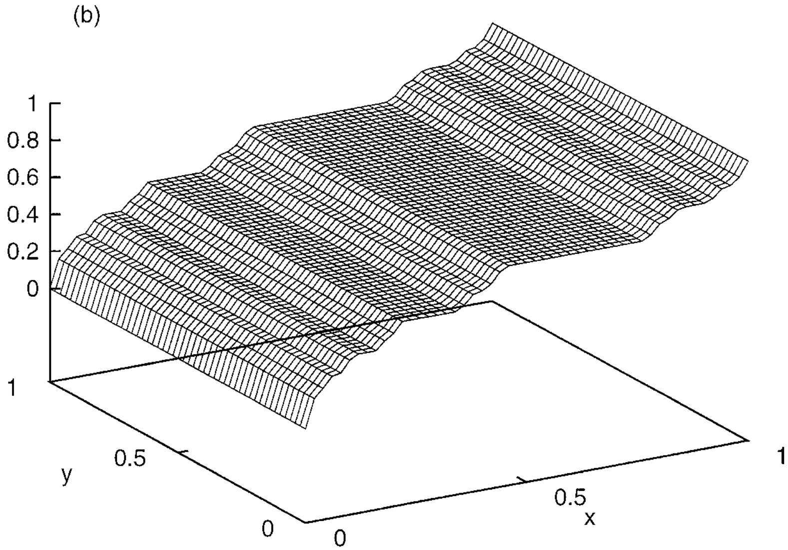


Fig. 2. (a) The derivative $\mu_x^F(x, y) = (\partial/\partial x)\, \mu^F([0, x] \times [0, y])$ of the conditionally invariant measure μ^F related to F which is specified by Eqs. (4), (5) and with choice $v(x) = d \cdot x(1-x)$, $R = 1.2$, $J^F = 0.7$, $d = 0.5$. (b) The derivative $\mu_y^G(x, y) = (\partial/\partial x)\, \mu^G([0, x] \times [0, y])$ of the conditionally invariant measure μ^G related to G which is specified by Eqs. (4), (5) and $v(x) = d \cdot x(1-x)$, $R = 1.5$, $J^G = 0.8$, $d = 0.5$.

Let us study now the iteration G with escape. Starting from $P_0(x, y) = 1$, after n iterations we obtain

$$P_n(x, y) = \begin{cases} (J^G)^{-n} & \text{if} \quad f^n(x) \in [0, 1] \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

Together with normalization by factor $\exp(\kappa^G n)$ this yields the density of the conditionally invariant measure μ^G in the limit $n \to \infty$, where κ^G is the escape rate. The x-dependence of $P_n(x, y)$ is, apart from normalization, equivalent to the restriction of the conditionally invariant density of the 1D map $f(x)$ to $f^{-n}[0, 1]$. So in the x direction $P_n(x, y)$ gives in the limit $n \to \infty$ after normalization the natural measure of the 1D map $f(x)$. Therefore the conditionally invariant measure of G is equal to the natural measure of the 1D map $f(x)$ in x direction and uniform in y direction. Figure 2b illustrates this measure. For reasons similar to case of map F the derivative $\mu^G_y(x, y) = (\partial/\partial y)\, \mu^G([0, x] \times [0, y])$ is displayed. When there is a density it is connected to μ^G_y as $P^G(x, y) = (\partial/\partial x)\, \mu^G_y(x, y)$. However, in the present case μ^G_y is a devil's staircase and there is no bounded density. Since the normalization factor in the natural measure of $f(x)$ is $\exp(\kappa^F n)$ the normalization factor for P_n is $\exp(\kappa^F n)(J^G)^n$ and the corresponding escape rate is $\kappa^G = \kappa^F - \log J^F$. This connection can be written in a symmetrized form as

$$\kappa^F - \tfrac{1}{2} \log J^F = \kappa^G - \tfrac{1}{2} \log J^G \tag{10}$$

Note that κ^G is zero if the equality sign applies in Eq. (6).

If we apply the above construction of' the natural measure to the map G we replace n by $-n$ in U_n and obtain the same set. Due to the constant Jacobian, starting with the Lebesgue measure we obtain the uniform distribution in U_n for the approximation of the natural measure, as in case of map F. Therefore their limits, the natural measures of F and G are identical.

It is interesting to study the special cases when one of the escape rates, κ^F or κ^G is zero. (Note, if both are zero the map is not dissipative, which case we do not want to discuss.) If $\kappa^F(\kappa^G)$ is zero the map $F(G)$ exhibits permanent chaos. Then the conditionally invariant measure $\mu^F(\mu^G)$ similarly to the natural measure becomes invariant under the mapping $F(G)$. This is a fractal measure (not smooth in the $y(x)$ direction) since $\kappa^G(\kappa^F)$ is still positive. On the other hand the iteration $G(F)$ maps a constant density to a constant one in the square U, therefore the conditionally invariant measure $\mu^G(\mu^F)$ is the Lebesgue measure in U.

4. CRITICAL STATE

We call a state critical if the invariant natural measure concentrates on a non-fractal subset of the repeller while the conditionally invariant measure is smooth along the unstable direction when one starts with the Lebesgue measure. In the examined 1D maps and in the present 2D ones this non-fractal subset is a fixed point. In 1D maps the borderline situation showing critical transient chaos is achieved when the slope of the map at a fixed point equals[19, 20] e^{κ}. In the representation, used here, when the Lebesgue measure is a conditionally invariant one this means that the slope of the map is infinity in the preimage of the fixed point, as can be seen from (3). Towards this limit while the conditionally invariant measure remains smooth the natural measure degenerates to a Dirac delta function at the fixed point.[19, 20] It occurs for maps (4), (5) for $v'(0)=1$. In this case the inverse map (5) behaves near $x=0$ as

$$f_u^{-1}(x)-1 \propto x^{\omega} \tag{11}$$

where $\omega > 1$.

In the 2D map (1) the condition for criticality, as will be shown below, is that the positive Lyapunov exponent of the saddle point in the origin agrees with κ^F. In terms of $v(x)$ for the map (1) with (4), (5) the condition again reads as $v'(0)=1$. We can see that at criticality two conditionally invariant measures μ_A^F, μ_B^F have to be considered possessing necessarily different escape rates.[7, 25] In case $v'(0)=1$ the side of $F\mathrm{A}_1$ at $x=0$ (which is the image of the $x=1$, $y\in[0,1]$ line segment) shrinks to a point, as seen from (1) and (5). Since the Jacobian is constant, necessarily the modulus of the local Lyapunov exponent becomes infinity in both directions. The behavior of the measure can be characterized by the measure of the rectangles $\mathrm{Box}(0,x)=[0,x]\times[0,1]$ and $\mathrm{Box}(x,1)=[x,1]\times[0,1]$. Integrating (8) over $\mathrm{Box}(0,x)$ one obtains

$$L\mu(\mathrm{Box}(0,x))=\mu(\mathrm{Box}(0,f_0^{-1}(x)))+\mu(\mathrm{Box}(f_1^{-1}(x),1)) \tag{12}$$

Therefore, starting with the Lebesgue measure, when $\mu(\mathrm{Box}(0,x))=x$, it follows from (3) and (12) that $L\mu(\mathrm{Box}(0,x))=e^{-\kappa}x$. Similarly, the property $\mu(\mathrm{Box}(0,x))\propto x$ is kept in further iterations, thereby the normalized conditionally invariant measure also possesses the property $\mu_A^F(\mathrm{Box}(0,x))=x$ with escape rate $\kappa^F=\kappa$. Consequently $\mu_{Ax}^F(x, y=1)=(\partial/\partial x)\,\mu_A^F(\mathrm{Box}(0,x))$ is unity for every x, as seen in Fig. 3a. On the other hand, inserting infinitesimally small x into (12) one obtains $L\mu_A^F(\mathrm{Box}(0,x))=f_l^{-1}(x)\,x$, therefore $\kappa_A^F=\log f'(0)$. Starting now with a smooth distribution with scaling $\mu(\mathrm{Box}(0,x))\propto x^{\omega}$ for small x due to (11) both terms in (12) scale

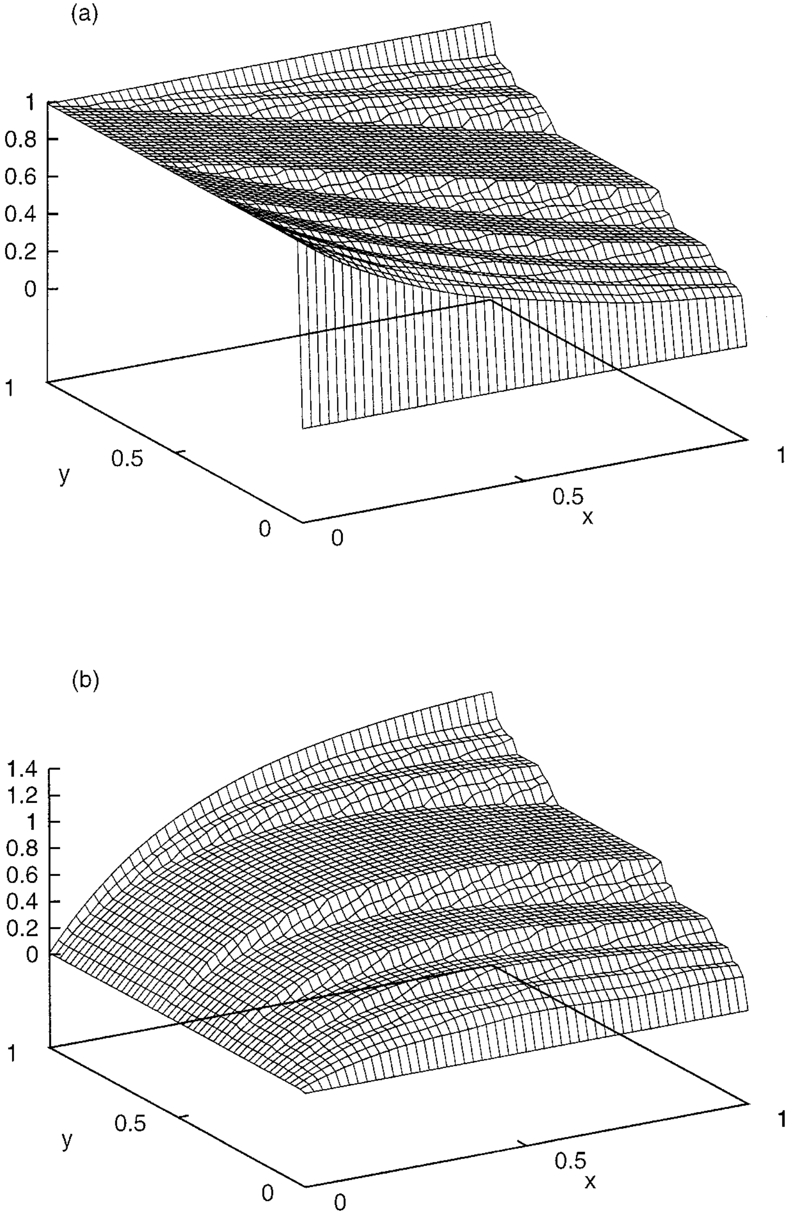


Fig. 3. The derivatives of the conditionally invariant measures in the critical state. The parameters are the same as in Fig. 2, except that $d=1$. (a) $\mu^F_{Ax}(x, y)=(\partial/\partial x)\,\mu^F_A([0, x]\times[0, y])$, (b) $\mu^F_{Bx}(x, y)=(\partial/\partial x)\,\mu^F_B([0, x]\times[0, y])$, (c) $\mu^G_y(x, y)=(\partial/\partial y)\,\mu^G([0, x]\times[0, y])$. The subscripts A and B refer to different initial distributions as defined in the text.

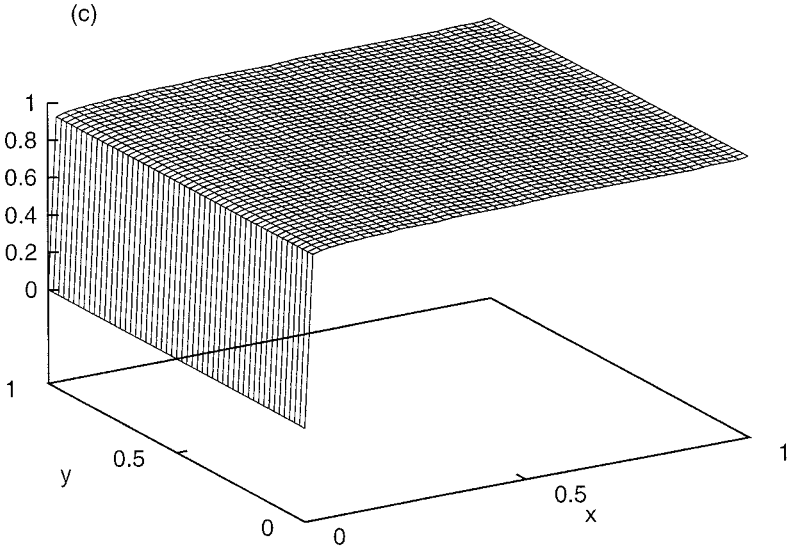


Fig. 3. (*Continued*)

as x^{ω}, leading to a conditionally invariant measure for which $\mu_B^F(\text{Box}(0, x)) \propto x^{\omega}$ and $\mu_{Bx}^F(x, y) = (\partial/\partial x)\, \mu_B^F([0, x] \times [0, y]) \propto x^{\omega - 1}$ for small x. Such a measure with $\omega = 2$ is shown in Fig. 3b.

Consequently any initial distribution with density vanishing fast enough at $x = 0$ (and not diverging too fast at $x = 1$) will approach μ_B^F when iterated keeping its norm constant. It can be shown, similarly as for the case of 1D maps,[25] that the condition is that the measure $\mu_B^F(\text{Box}(0, x))$ decreases at least as fast as x^{σ_c} when $x \to 0$ where $\sigma_c = \kappa_B^F/\kappa_A^F$ and $\mu_B^F(\text{Box}(x, 1))$ decreases at least as $(1 - x)^{\sigma_c/\omega}$ when $x \to 1$.

It is more surprising, that the criticality has serious consequences also for the characteristics of the iteration G. This appears already in the form of the conditionally invariant measure μ^G of the map G obtained starting with the Lebesgue measure, when approaching criticality. In this limit its density becomes a Dirac delta function in x and constant in y direction, as it follows from the considerations of the previous section (see Fig. 3c). Though the natural measure corresponding to μ_B^F is also invariant under action of G, there is no counterpart of μ_B^F for the map G. Instead, as it can be easily checked, at the critical state there is an infinity of conditionally invariant measures μ_η^G smooth in the y direction. They are concentrated on the $x = 0$ line segment, similarly to μ^G. Namely, they have the density $y^{\eta} \delta(x)$ with $\eta > -1$, and the escape rates $\kappa_\eta^G = (\eta + 1)\, \kappa^G$ belongs to them.

Among them the case $\eta=0$ represents the conditionally invariant measure obtained in the limit of criticality. Initial measures that are smooth in U with nonzero value outside the $x=0$ line belong to the basin of attraction of μ^G, and the initial measures with densities $\phi(y)\,\delta(x)$ with $\phi(y)=\mathcal{O}(y^\eta)$ for $y \ll 1$ belong to that of μ^G_η. Note that when deriving Eq. (10) it was assumed that we start with the Lebesgue measure. The condition for criticality is again that the escape rate κ^G equals to the positive Lyapunov exponent of the saddle point at the origin, which in case of map (7) with (4), (5) is $\log(RJ^G)$.

Let us turn now to the natural measures. Still assuming that κ^F and κ^G are positive it can be seen that the natural measure corresponding to μ^F_A becomes degenerate similarly as in the 1D map, namely, its density is $\delta(x)\,\delta(y)$. This follows from the fact that this natural measure is dominated by the trajectories getting close to the saddle point in x direction. Since they spend most of their lifetime in that vicinity and the saddle is attracting in y direction with the rate $\kappa^F - \log J^F = \kappa^G$ they are very close to the saddle in most of their lifetime also in y direction. Numerical approximation of the density of this natural measure, presented in Fig. 4a, also shows this concentration to the fixed point at the origin, while the repeller itself does not degenerate in a similar way (see Fig. 4b). Conversely to μ^F_A, (both in $\kappa^G=0$ and $\kappa^G>0$ cases) the natural measure corresponding to μ^F_B is distributed on the whole invariant set except along the line $x=0$.

Finally we briefly discuss the cases when one of κ^F and κ^G is zero. In case of $\kappa^G \to 0$ the measure μ^G conditionally invariant under the iterations by G becomes truly invariant and the related natural measure coincides with it in the limit. The behavior of the natural measure can be seen by noticing that in case $\kappa^G=0$ there is no attraction in y direction at $x=0$. The fixed point at $x=0$, $y=0$ and every other point of the line segment at $x=0$ becomes a marginal fixed point (they form a fixed line). Since in the present case $F^n \mathrm{U} \cap \mathrm{U} = \mathrm{U}$ and $F^{-n}\mathrm{U} \cap \mathrm{U}$ is uniform in y direction the natural measure concentrated in $x=0$ is distributed in y direction evenly, similarly to the conditionally invariant measure of G at $\kappa^G>0$. Consequently the density of the measure invariant under G is $\{\delta(x); 0 \leqslant y \leqslant 1\}$. The dynamics generated by G is strongly intermittent, which is, however, unusual in the sense that the measure is not concentrated into a fixed point or a periodic orbit but to a line segment. It means that at the intermittent recurrences of the "regular motion" the values of variable y are evenly distributed between 0 and 1, while values of x are always near zero then.

In the limit κ^F goes to zero one has to follow the development of the measures of type A and B, as well. One can easily convince oneself, that the measure μ^F_A remains unaltered but, of course, becomes invariant. The related natural measure (entirely concentrated in the fixed point at the origin) looses its connection to μ_A^F when $\kappa^F \to 0$, i.e., the measure concentrated in the fixed point can not be reached by iterations starting with a smooth measure. Furthermore μ_B^F (and the natural measure related to it) becomes identical to μ_A^F in this limit. This measure represents a weakly intermittent dynamics.

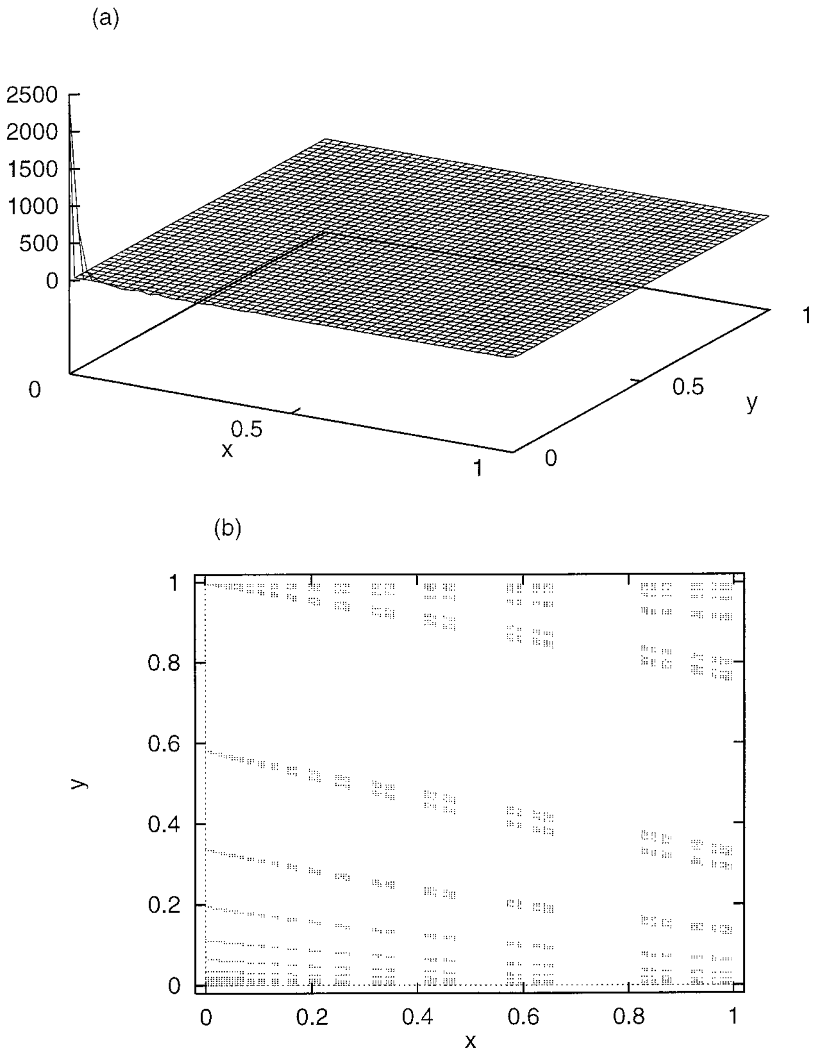


Fig. 4. Further numerical results in the critical state for the same map and parameters as in Fig. 3a: (a) the approximate density of the natural measure corresponding to μ_A^F, (b) the repeller in finite resolution.

5. TRANSIENT DIFFUSION

If escape is possible typical long trajectories show a chaotic motion in a finite duration after which they escape. The chaotic time period is unlimited and the behavior of long trajectories is determined by an invariant set. In the extended system the transiently chaotic motion leads to transient diffusion. Such transient diffusion has been studied in one-dimensional maps and it has been shown that nonlinearity has strong effects on it.[7]

We turn now to investigate the transient diffusion in the framework of dynamics generated by F and G, taking into consideration the variable S in (1) and (7). Since the diffusion coefficient is defined in the limit of infinitely long trajectories it is an average taking the initial values according to the natural measure ν. It is given by[7, 26]

$$D = \lim_{t \to \infty} \frac{\langle (S_t - \langle S_t \rangle_\nu)^2 \rangle_\nu}{2t} \tag{13}$$

where $\langle S_t \rangle_\nu$ is the average drift and we start with $S_0 = 0$.

First consider the noncritical situation, when ν is obtained starting with the Lebesgue measure. Choosing first the map F and starting with a point (x_0, y_0) from the cell $S_0 = 0$ we obtain

$$(x_t, y_t, S_t) = F^t(x_0, y_0, 0) \tag{14}$$

Since ν is invariant distributing initial points (x_0, y_0) according to it one observes the same distribution for (x_t, y_t). Obviously

$$G^t(x_t, y_t, S_t) = (x_0, y_0, 0) \tag{15}$$

and due to translational invariance

$$G^t(x_t, y_t, 0) = (x_0, y_0, -S_t) \tag{16}$$

Furthermore recalling the result in Section 4, that ν is the same for F and G, one concludes that the definition (13) gives the same diffusion coefficient for the maps F and G.

Let us turn now to the critical state. We have seen that in case of map F we have to consider two natural measures. The first one is related to the Lebesgue measure and is entirely concentrated into the fixed point. Consequently, the diffusion coefficient is zero. The other one leads, however, to a nonzero diffusion coefficient.

Regarding the map G the natural measure related to μ^G leads also to $D=0$. That means changing a control parameter the diffusion coefficient tends to zero when approaching criticality. For the map F a jump can occur in D when reaching criticality, while for G this is not the case.

6. SUMMARY AND DISCUSSION

In this paper we have followed the development of conditionally invariant and related natural measures when altering a control parameter of the generalized multibaker map. It has been investigated which conditionally invariant measure we arrive at if the Lebesgue measure is the initial one. It can be easily seen that away from the critical state (i.e., $v'(0)<1$) the same conditionally invariant measure is obtained by starting the iteration with such a measure that decreases when approaching the line segments $x=0, 1$, $0 \leqslant y \leqslant 1$ at least as fast as the Lebesgue measure (the condition can be weakened, so the complete basin of attraction is larger). We call a conditionally invariant measure typical if its basin of attraction is specified by limitation of its scaling only from above at both of the above mentioned line segments, thereby its basin of attraction is large with respect to its exponents at $x=0$ and $x=1$. In particular a typical conditionally invariant measure has such smooth distributions in its basin of attraction that are concentrated in the interior of the unit square. In this sense the conditionally invariant measure developed from the Lebesgue measure is typical away from the critical state. At the critical state, however, this is no more the case if we treat the map F. Starting with the Lebesgue measure one arrives at μ_A^F, but its basin of attraction is restricted in the way that the initial measure should start linearly at $x=0$ ($0 \leqslant y \leqslant 1$) if it behaves at $x=1$ as $\mu^0(1)-b(1-x)^\tau$, $\tau>1/\omega$ or it can have more general scaling possibilities at $x=0$ if however its asymptotics at $x=1$ ($0 \leqslant y \leqslant 1$) is fixed. So according to our definition above μ_A^F is not typical. One should point out that there are continuously many non-typical conditionally invariant measures at and away from the critical state, as well, whose discussion, however, is beyond the scope of the present paper. In case of 1D maps these measures have been investigated in detail in ref. 25. Their appearance is connected to the existence of singular eigenfunctions of the Frobenius–Perron operator first pointed out and studied by MacKernan and Nicolis in case of piecewise linear maps.[27] The measure μ_A^F is unique among the other non-typical measures at the critical state because it is the limit of typical ones when approaching the critical state by changing the control parameter. Furthermore this conditionally invariant measure is obtained when one starts with the Lebesgue measure. At the critical situation μ_B^F is typical, its basin of attraction contains distributions specified in Section 4. Summarizing, if one changes the control parameter the following behavior emerges in case of the map F starting with a distribution with density concentrated in the interior of the unit square: away from the critical state the measure approaches the same conditionally invariant measure as the Lebesgue measure while arriving at the critical state the measure μ_B^F is obtained, which does not have the Lebesgue measure in its basin of attraction. This leads to a jump in the diffusion coefficient when reaching the critical state.

Concerning the map G, away from the critical state the same properties apply as for F. The typical conditionally invariant measure remains typical, however, in the limit of the critical state with the basin of attraction containing distributions which are not concentrated on the $x = 0$, $0 \leqslant y \leqslant 1$ line segment. Among the non-typical conditionally invariant measures existing for the map G we mentioned in Section 4 those which are concentrated on the line segment mentioned above. The behavior of the typical conditionally invariant measure has led to a vanishing diffusion coefficient when approaching the critical state in case of the map G.

There is an aspect of our investigation which we would like to emphasize. It is generally assumed in case of transient chaos that the initial measure is the Lebesgue one. It is then a basic question, what is the basin of attraction of the conditionally invariant measure which is arrived at when one starts with the Lebesgue measure. Our results show that this conditionally invariant measure has a considerable basin of attraction in general, an exception is provided by the critical state in case of map F. Finally one has to point out that other initial measures outside the basin of attraction of this conditionally invariant measure might be realized not only in numerical, but also in laboratory experiments.

Taking the limit of permanent chaos along the "thermodynamic path" maintaining the critical state we have discussed the intermittent behavior of the dynamics and pointed out also unusual properties. The intermittent dynamics can be contrasted to that generated by the map F with the μ_A^F measure in the critical state. As shown in Section 4 the related invariant (natural) measure is entirely concentrated in the origin, a feature shared also by strong intermittency in its usual form, which means that in both cases the trajectories spend the majority of time near the origin. The transient chaos corresponding to the μ_A^F measure is qualitatively different, however, from intermittency (at least not very close to the permanent chaos limit) since the majority of trajectories go away from the vicinity of the origin without exhibiting regular-like sequences characteristic to intermittency and they escape.

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